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Connectivity Properties of Continuum Percolation Processes on $\mathbb{R}^2$

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A new approach is introduced in order to estimate the critical parameters of continuous percolation of overlapping disks in $\mathbb{R}^2$ when the centres of the disks are Poisson distributed. Better insights on relevant parameters near criticality are found. Moreover, introducing a suitable connectivity length, the model is able to describe continuous percolation of any type of geometrical objects in $\mathbb{R}^2$.

KEY WORDS: Continuum percolation; k-clusters; critical density.

1. INTRODUCTION AND DEFINITIONS

The morphology of disordered systems has two major aspects: topology describing the connectivity of the microscopic elements of the system and geometry taking into account the shapes and sizes of these individual elements. Percolation tells us when a disordered system is macroscopically connected.

In Mathematical Physics, percolation theory on regular lattices has already a long standing and successful history. On the contrary percolation in continuous systems and on random graphs has received less attention in spite of its great interest in applications. Over the past four decades, percolation

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has been applied to modeling a rich variety of phenomena in disordered systems like conductivity in disordered semiconductors, permeability of porous media [FH], fracture network of rocks [Sah], spread of Aids epidemics [Bl] and even quark-gluon plasma formation in QCD gauge theories [Sa].

We would like to stress that our approach here will refer to continuous systems without introduction of any lattice approximation.

Let us recall the main features of continuous percolation, i.e., modeling a random medium—in fact randomly distributed objects in homogeneous media—by using the features of stochastic geometry. The basic facts are quite elementary. Let $\{x_i\}$, $i \in I \subset \mathbb{N}$ the points of a standard Poisson process in $\mathbb{R}^d$ $(d > 1)$ endowed with the Euclidean distance, with intensity ρ.

For each x_i one chooses at random a geometrical object from a set $\mathfrak{A}$ and places it in $\mathbb{R}^d$ so as to cover x_i. In the following we are mostly concerned with the cases in which $\mathfrak{A}$ is a one point set (i.e., we have copies of one and the same geometrical object $\mathscr{P}$, typically a sphere of given radius R). Moreover $\mathscr{P}$ has a distinguished point X (and sometimes a preferred orientation) and the ith copy of $\mathscr{P}$ is placed so that $x_i = X$.

Denoting by $\mathscr{P}_i$ the object centered in x_i we are interested in clusters of overlapping objects and in particular in the existence of a critical value of ρ such that for $\rho \gtrsim \rho_c$ there exists an infinite incipient cluster "spanning" all the space domain. By this we mean as usual that for any ρ larger than ρ_c, with probability one, there is a sequence of cubes K_i of side $\mathscr{L}_i$ with $\mathscr{L}_i$ going to infinity as i goes to infinity, such that for any i, there is a connected cluster which "spans" K_i, i.e., touches opposite sides of K_i.

Let declare two objects $\mathscr{P}_i$ and $\mathscr{P}_j$ *adjacent* (or neighbour) if $\mathscr{P}_i \cap \mathscr{P}_j \neq \emptyset$. We say that two objects (or the two corresponding points x_i and x_j) are connected, $\mathscr{P}_i \leftrightarrow \mathscr{P}_j$ (or $x_i \leftrightarrow x_j$), if there exists a chain of adjacent objects (or neighbour points) connecting these two objects (or these two points). A *cluster* is a maximal set $\{\mathscr{P}_i\}_{i \in J \subset \mathbb{N}}$ (or $\{x_i\}_{i \in J \subset \mathbb{N}}$) of connected objects (or points). The aim of continuous percolation theory is to study the sizes and shapes of clusters for specified values of ρ and typical objects $\mathscr{P}$. As in the case of lattice percolation in a infinite domain, the existence of an "infinite incipient" cluster is linked to the long range connectivity property of the random system. The study of the phase transition associated with the appearance of an infinite cluster is one of the main problem of percolation theory. Objects as diverse as disks, squares, needles in $\mathbb{R}^2$, spheres, cubes, ellipsoids, in $\mathbb{R}^3$ have been used for typical models (see, e.g., [G, PS]).

For general results on continuous percolation see [Ba, Al1, Al2, I]. For recent results concerning the scaling properties of the infinite percolating cluster see [A1, A2]. For links between continuous percolation and phase transitions in spins models see, e.g., [C1, C2]. Percolation theory in continuous media is naturally related to classical problems of stochastic geometry, see, e.g., [St].

In the following section we state the framework in which we want to develop our approach and give a model for the formation of clusters of typical sizes and shapes. We end this section by giving the expected relation between the (translation invariant) model in $\mathbb{R}^2$ and the result of numerical simulations which deal with finite systems and a finite number of outcomes.

To take a definite example, we consider in $\mathbb{R}^2$ a distribution of disks of radius R thrown at random with mean density ρ in a large volume V which we assume to be a square of side $\mathscr{L}$ $(V = \mathscr{L}^2)$. The disks are allowed to overlap.

The mean density ρ is the relevant physical parameter describing the system, but it is more convenient to use as equivalent dimensionless parameter the filling factor $\eta = \rho\pi R^2$ which represents the mean number of centres on the surface of any given disk. When η is very small, for a "typical realization ω," the vast majority of the disks are isolated and the clusters are very small. When η increases, the size of the clusters increases and at a given value $\eta_c(V, \omega)$ there is a connected cluster which joins opposite sides of the square.

If the random variable $\eta_c(V, \omega)$ converges in probability law, when $\mathscr{L}$ goes to infinity, to a constant η_c, we say that when $\eta \gtrsim \eta_c$ there is an incipient spanning cluster. The convergence of $\eta_c(V, \omega)$ to a constant η_c is expected to follow a one-zero law.

The constant η_c has been obtained by numerical simulations. For a system of overlapping disks one obtains η_c nearly 1.18 (see [I] and references therein).

The existence of an infinite cluster is related to the connectivity property of the random system. The main characteristics of the cluster distribution are:

— the probability $\mathbb{P}_\infty$ for a disk to belong to an infinite cluster

— the correlation length ξ which is the maximum size above which the clusters are exponentially scarce

— the average cluster size S_{cl} (mean size of the clusters containing one fixed disk).

There is a strong numerical evidence that, for different types of percolation (on various lattices, for both bond or site percolation) and in the limit of infinite volume, there exists, near the percolation threshold, a power law dependence for the quantities $\mathbb{P}_\infty$, ξ, S_{cl}, with characteristic critical exponents β, ν, γ such that:

$$
\begin{aligned}
\mathbb{P}_\infty(\eta) &= \left(1-\frac{\eta_c}{\eta}\right)^\beta; \qquad \eta \geqslant \eta_c \\
\xi(\eta) &= |\eta-\eta_c|^{-\nu} \\
S_{cl}(\eta) &= |\eta-\eta_c|^{-\gamma}.
\end{aligned}
\tag{1.1}
$$

These critical exponents are believed to reflect a universal property of percolation and to depend only on the dimension d and not on the structural details of the lattice (square, triangular,...) nor on the type of percolation (site, bond or continuous). This universality is expected to be a general feature of second order phase transitions.

In two dimensions these exponents are conjectured to be exact (see [Bax], [D] and [Gr, p. 252]):

$$\eta_c \approx 1.18, \qquad \beta = 5/36, \qquad \nu = 4/3, \qquad \gamma = 43/18.$$

In [ZS] a (not one-to-one) mapping from continuous to lattice percolation is derived allowing to prove the existence of a non-trivial critical percolation density in the continuous case (see also [Gr]). In [MR], a systematic account of continuous percolation is given, in particular a rigorous approach enables the derivation of bounds for the critical density of disks in two dimensions.

The same description can be used for sphere percolation in $\mathbb{R}^3$ with a filling factor $\eta^{(d=3)} = \rho \frac{4}{3} \pi R^3$. In this case the critical exponents have been, up to now, obtained uniquely by numerical simulations. The values found here are:

$$\eta_c \approx 0.35, \qquad \beta \approx 0.41, \qquad \nu \approx 0.90, \qquad \gamma \approx 1.8.$$

Nevertheless, in $\mathbb{R}^3$, as in $\mathbb{R}^2$, lattice percolation and continuous percolation belong to the same universality class of critical phenomena: the class of random (uncorrelated) percolation in $\mathbb{R}^d$.

Remark 1.1. In both dimensions 2 and 3, the fraction Φ of volume occupied by randomly overlapping objects is defined by $\Phi(\eta) = 1-\exp(-\eta)$ (see [SK]) and $\Phi(\eta_c)$ is found less than η_c

$$
\begin{aligned}
\Phi(\eta_c) &\approx 0.68 \qquad \text{in} \quad d=2, \\
\Phi(\eta_c) &\approx 0.30 \qquad \text{in} \quad d=3.
\end{aligned}
$$

In $\mathbb{R}$ the continuous percolation problem is trivial. Indeed, $\Phi = 1$, which requires $\eta = \infty$ and $\rho = \infty$.

However, besides numerical simulations there is till now no approach to calculate the critical parameters and the typical processes underlying continuous percolation as was done [K] for bond percolation on the lattice $\mathbb{Z}^2$. Our aim in this work is to develop a strategy leading to an estimation of the critical parameter η_c and to a better insight concerning the sizes and shapes of the typical clusters near the percolation threshold.

Our procedure for overlapping disks is described in Section 2. In Section 3, we show that the approach is quite generic in the sense that it can be adapted to study continuous percolation of objects of any shape. Several examples are given in Section 4.

2. CLUSTER MODELS

In [Al1] an heuristic percolation criterium for continuous systems is proposed based on the comparison of two fundamental lengths of the system, namely the average bounding length l_1 (defined as the mean distance between the centres of two adjacent disks) and the average distance l_k between the centres of two adjacent disks each of which having at least k neighbours. The authors of [Al1] postulated that percolation, which can be regarded as the condition that the system be macroscopically connected, occurs when the condition $l_k = 2l_1$ is fulfilled for some definite value of k depending on the space dimension.

They approximated the mean distance l_k of points having at least k neighbours using their effective density, which can be computed from the basic Poisson law. The results they obtained ($k=4$ in dimension 2 and $k=2$ in dimension 3), are in good agreement with numerical simulations. The aim of our work is to use probabilistic arguments to improve on [Al1] and give an insight concerning the size and structure of the clusters near the percolation threshold.

We will mainly concentrate on a case in which the objects are in $\mathbb{R}^2$ and have the same size and shape, namely disks of radius R. In a box of size $\mathscr{L}$ we throw N disks at random with density $\rho = N/\mathscr{L}^2 = N/V$. A k-cluster γ_k is a maximal set of k overlapping disks; its shape and diameter $D(\gamma_k)$ depend on the positions of the centres of the disks (the diameter is the largest distance between two points lying on any of the disks of the cluster).

We shall give below arguments that support the following picture: if $\rho < \rho_c$ for a box of size $\mathscr{L} \gg R$ and for a vast majority of realizations, the relative number of k-clusters is small if $k \geqslant k_{\max}$; we predict $k_{\max} = 6$ or 7, which is compatible with a critical density $\eta_c \simeq 1.18$ and also with numerical simulations (of course if ρ is very small the k-clusters are scarse if $k \geqslant 2$).

When two clusters γ_k and $\gamma'_{k'}$ overlap, they form a $(k+k')$-cluster. Our arguments suggest that this happens only at a density $\rho < \rho_c$, $\rho \simeq \rho_c$ (i.e.,

$\rho = \rho_c - \epsilon$ for ϵ very small) and that it happens "simultaneously" at all length scales (i.e., no matter how large is $\mathscr{L}$). In other words, by increasing the density (i.e., by adding a few disks per unit volume), the incipient spanning cluster emerges "suddenly" from configurations in which there is an overwhelming majority of small connected clusters whose distance goes to zero as $\rho \uparrow \rho_c$. This suggests that the "scale-invariance" argument, typical of a renormalization-group procedure, is valid even at the scale of very small clusters.

For a box of size $\mathscr{L} \gg R$ and for a given realization $\omega(N, \rho)$ of the process, denote by the name of "global k-cluster" the collection of the k-clusters in the realization. Notice that for any realization ω this defines an (empirical) measure $\mu_k^{N,\rho}$ on the set of k-clusters, for every N. For each fixed value of the density ρ, in the limit $N \to \infty$, $V \to \infty$, $N/V = \rho$, this defines an empirical measure μ_k^{ρ} on the k-clusters, for each value of $k = 1, 2, \ldots$; the measure is (almost surely) independent from the sequence of realizations chosen.

We shall denote by μ^{ρ} the induced measure on all clusters and by D_k the average under μ_k^{ρ} of the diameter of the k-clusters.

Describing the measure μ is a difficult task; for a given realization in a volume V there may be big fluctuations in $\mu_k^{N,\rho}$ as N varies (adding a few disks for a given finite $\mathscr{L}$ may change discontinuously the average of n_k/N by changing two clusters into a bigger one, n_k being the number of k-clusters). Therefore we will later switch our attention to a different measure, $\tilde{\mu}^{\rho} = \sum \tilde{\mu}_K^{\rho}$, where $\tilde{\mu}_K^{\rho}$ is concentrated on a different set of configurations which we shall denote by the name "spherical configurations of $K+1$ disks" and for short by "spherical $(K+1)$-clusters" (see Section 2.2).

We shall see at the end that, for the problem we are considering, the set of configurations on which $\tilde{\mu}$ is concentrated gives a nice description of the properties of the system (known from numerical simulations) and can then be considered as leading in the mean to the same results as the real measure μ.

The measure $\tilde{\mu}_K^{\rho}$ depends only on a free parameter with the dimension of a length (the radius by which one will define the spherical clusters) that we will eventually connect with the distribution of the radii in the global cluster. For the sets on the support of $\tilde{\mu}_K^{\rho}$, it is comparatively easy to give a definition of connectedness in term of an effective distance d between the centres of two spherical clusters. This leads to a determination of the critical density and at the same time gives a relative probability of spherical $(K+1)$-clusters near the percolation threshold in fair agreement with the available numerical simulations.

Actually, relying on numerical simulations showing random configurations of discs near criticality, one can see that the main part is made of compound structures of few discs. These compound structures can be seen as the building blocks of the general clusters.

Let us now introduce the main ideas on which we base our argument. Considering independent families of spherical $(K+1)$-clusters (with a fixed number of elements) our aim is to establish whether some families are more relevant than others at the percolation threshold. This leads simultaneously to the determination of the critical density ρ_c. This idea can also be related to several kinds of cluster models already investigated in the past, e.g., the "nodes, links and blobs model" of Pike and Stanley [PSt] where related type of arguments were introduced in order to characterize the global features of clusters near criticality.

It should be noticed that the mechanism we propose provides at the same time a rough idea of the shape of the infinite cluster, since the specific realizations of the involved clusters possess a structure suggesting the predominance of prongs which attach to each other leaving holes in between.

A further remark concerns the possibility to have a unified treatment for different types of shapes of the elementary objects, provided that their distribution preserves the translational and rotational invariances. In that case we have only one typical length (depending on the shape and size of the elementary objects), the "connectivity distance" a between the distinguished points of the elementary objects. One can then introduce spherical clusters whose critical radius as well as the effective distance between the centres of two clusters are expressed in a standard way in term of a and exactly the same approach as before leads to the specific critical density.

2.1. Characteristic Size of the *k*-Clusters

First let us collect some results concerning the cluster sizes.

In $\mathbb{R}^2$ we consider disks with a fixed radius R distributed with a uniform density ρ. Disks are then defined by their centres, say O_i, $i \in I_k = \{1,..., k\}$. Two disks O_1, O_2 are neighbours iff $|O_1 O_2| \leqslant 2R$. The underlying probability is the Poisson law. The probability that there are k occupied points in a domaine of surface area v is given by

$$\mathbb{P}_k(\rho v) = \frac{(\rho v)^k}{k!} \exp(-\rho v) \tag{2.1}$$

and ρv is the mean number of occupied points in the domain.

Consider a cluster of size k (group of k connected disks). A sufficient condition in order to take into account that the cluster contains exactly k elements is to consider that it is surrounded by an empty region whose minimal volume depends on the shape of the cluster. The volume of this

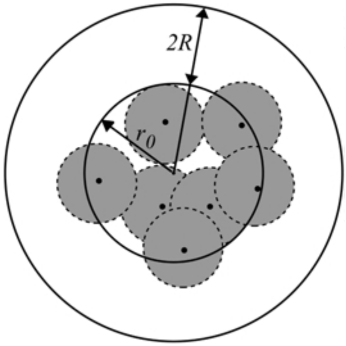

Fig. 1. k-cluster of radius r_0.

empty region is minimal if the centres of the disks almost coincide, but this event is globally rare. For the same total volume of overlapping disks, the empty volume is minimal if the disks are brought together in a compact shape.

In this spirit, let us consider the following events

$$\mathscr{A}_1 \underset{\text{def}}{=} \{\text{there are } k \text{ centres in a sphere of radius } r_0\},$$

$$\mathscr{A}_2 \underset{\text{def}}{=} \{\text{there is no centre in the corona }]r_0, r_0+2R]\}.$$

In $\mathbb{R}^d$ the probability of the event $\mathscr{A}_1 \cap \mathscr{A}_2$ is given by: (with π_d denoting the volume of the unit ball in dimension d):

$$\begin{aligned}&\mathbf{P}(k\text{-cluster of radius } r_0)\\ &\quad= \mathbf{P}(\mathscr{A}_1 \cap \mathscr{A}_2) = \frac{(\rho\pi_d r_0^d)^k}{(k)!} \exp(-\rho\pi_d r_0^d) \exp[-\rho\pi_d((r_0+2R)^d - r_0^d)].\end{aligned}$$

This probability is maximal for $r_0 = \tilde{r}_0(k)$ given by $\tilde{r}_0(\tilde{r}_0+2R)^{d-1} = \frac{k}{\rho\pi_d}$. For $d=2$, this leads to

$$\frac{\tilde{r}_0(k)}{R} = -1 + \sqrt{1+\frac{k}{\rho\pi R^2}} = -1 + \sqrt{1+\frac{k}{\eta}}. \tag{2.2}$$

2.2. The Spherical Cluster Model

The previous arguments give a hint as to which is the most probable radius of an element of the global k-cluster. Due to the invariance of $\mathscr{A}_1 \cap \mathscr{A}_2$ under rotation, the density of the centres in a global k-cluster is invariant under simultaneous rotation. To determine the actual joint distribution of k-clusters in a global k-cluster is still a formidable task. We introduce therefore a subfamily of the global k-clusters which is easier to handle and still suitable to describe in a proper way the critical parameters of the percolation transition.

We call "spherical $(K+1)$-cluster" centered in O the collection of configurations in which an occupied point O is surrounded by K occupied points $A_1,\ldots, A_K$ located on a circle of radius lR centered in O. The measure on this set of points is the one induced by the uniform distribution of each A_i on the circle of radius lR centered in O.

Remark 2.1. The parameter l above is for the moment a free parameter meant to take the place of the joint distribution of the $K+1$ centres of the disks in a global cluster. It is meant to characterize the mean distance between the centre of a "spherical $(K+1)$-cluster" and its K neighbours. This will be used to give a criterium by which we can state that two spherical clusters are separated from each other by looking at the distance of their centres. Eventually we shall identify lR with the "most probable" radius of the global $(K+1)$-cluster previously called $\tilde{r}_0(K+1)$.

For spherical clusters it is relatively easy to provide a definition of connectedness. Consider two spherical $(K+1)$-clusters centered in O_1 and O_2 respectively.

Let $|O_1 O_2| = \delta R$. The mean of the minimum square Euclidean distance between elements of two distincts spherical $(K+1)$-clusters is

$$[M_K(l, \delta)\, R]^2 = \langle \min_{(X,Y)} \{d^2(X, Y);\, X \in \{O_1, A_i\}_{i \in I_K},\, Y \in \{O_2, B_j\}_{j \in I_K}\}\rangle \tag{2.3}$$

where the average $\langle\ \rangle$ is w.r.t. the uniform distribution of the centres of the disks on the circle of radius lR.

We say now that two spherical $(K+1)$-clusters are connected if $M_K(l, \delta)\, R$ is less than or equal to $2R$. The critical value δ_c^K for connectivity is then defined by the equality

$$M_K(l, \delta_c^K) = 2. \tag{2.4}$$

The functions $M_K(1, \delta_c^K/l)$ can be determined and δ_c^K, which is a function of both K and l, satisfies the threshold condition corresponding to the percolation condition

$$M_K(1, \delta_c^K/l) = 2/l. \tag{2.5}$$

2.3. Connection with the Critical Density of Disks

In order that the "spherical $(K+1)$-clusters" give rise to percolation, their number (which is a function of the density of disks ρ) has to be high enough. If n_K is the density of such clusters (which we take to be uniformly distributed due to the uniform distribution of disks), the mean distance between the centre of two adjacent spherical $(K+1)$-clusters is nothing but $\sqrt{1/n_K}$. We call RL_K this mean distance.

We have now to precise what the density n_K is. In fact we shall use an approximation tied to the basic Poisson law describing the distribution of disks. Consider a disk centered in O. The probability that this disk intersect exactly K other disks is the conditional probability that in a circle with centre O and radius $2R$ there are $K+1$ occupied points one of them being O. It is given by (see formula (2.1))

$$\mathbb{P}_K(\pi(2R)^2\,\rho) = \mathbb{P}_K(4\eta). \tag{2.6}$$

The density of these configurations is just

$$\Xi_K = \rho\mathbb{P}_K(4\eta).$$

We assume that n_K is of the order of Ξ_K in such a way that the mean distance between the centres of the family of clusters we are interested in is given by the relation

$$RL_K \approx \sqrt{\frac{1}{\Xi_K}}. \tag{2.7}$$

The last hypothesis we make is that we can replace, at least near the percolation threshold, the distribution of disks described by the global k-clusters for fixed k by a distribution of spherical $(K+1)$-clusters $(k = K+1)$ with density Ξ_K. Percolation through "spherical $(K+1)$-clusters" occurs if the distance between the centres of two such clusters is, in the mean, less than or equal to the distance of connection $R\delta_c^K$. Near criticality, the following condition has therefore to be fulfilled

$$\delta_c^K = L_K. \tag{2.8}$$

The three relations (2.5), (2.7), and (2.8), together with the additional condition $lR = \tilde{r}_0(K+1)$ allow then to calculate the critical value $\rho_c(K+1)$ at which the spherical clusters of type $K+1$ percolate. We then obtain the percolation critical value at which percolation is expected to occur if only clusters of exactly $K+1$ elements are relevant. In fact considering the different families as independent, the expected critical density is the minimum with respect to K of $\rho_c(K+1)$

$$\rho_c = \min_K \rho_c(K+1).$$

In this expression, the value $K_{\min}$ of K corresponding to ρ_c leads to the expected informations concerning the shape of the dominant clusters near criticality.

Before proceeding to explicit calculations, we summarize the sequence of steps leading to a reasonable numerical value for the critical density for continuum percolation of uniformly distributed disks. At the same time it gives hints concerning the distribution of connected clusters near the percolation threshold. The only relevant parameters are the disk radius R and their density ρ.

1. Existence of a characteristic size for the "most probable" k-clusters at fixed density ρ. An isolated k-cluster is a set of k disks whose centres are located in a circle of radius r_0 surrounded by a void corona of thickness $2R$. This most probable k-cluster have then a radius $\tilde{r}_0(k) \equiv \tilde{r}_0(k, R, \eta)$ with $\eta = \pi R^2 \rho$, see formula (2.2).

2. Introduction of the set of configurations that we call "spherical $(K+1)$-clusters" which contain a parameter, with a dimension of a length, which we identify with $\tilde{r}_0(K+1)$, see Section 2.2. The spherical $(K+1)$-clusters are supposed to have a density which depends only on the basic Poisson law.

3. Connectivity condition between two spherical $(K+1)$-clusters $\mathscr{C}_1$ and $\mathscr{C}_2$. This condition is the requirement that the average minimum distance between the centres of two disks belonging, respectively, to $\mathscr{C}_1$ and $\mathscr{C}_2$, should be less than or equal to $2R$. When it is equal to $2R$, the distance δR between the centres of the clusters $\mathscr{C}_1$ and $\mathscr{C}_2$ reaches a critical value $\delta_c^K R$, see formula (2.5).

Percolation is expected to occur when the density ρ of disks is sufficient to insure that the connectivity condition is satisfied, i.e., that the mean distance between the $(K+1)$-clusters is of the order of $\delta_c^K R$, see formulas (2.7) and (2.8).

These ideas can be considered as independent and they all contribute to the characterization of the structure of the clusters at the percolation threshold and give an insight about the structure of the incipient spanning cluster.

2.4. Explicit Calculations ($d=2$)

In what follows, we will use dimensionless variables. If one identifies l with the radius of the "most probable" k-cluster containing $k=K+1$ disks, see Eq. (2.2), one gets:

$$l \equiv \frac{\tilde{r}_0(K+1)}{R} = z = -1 + \sqrt{1 + \frac{K+1}{\eta}}.$$

Denoting by $f_K(x) = [M_K(1, x)]^2$ (with $x = \delta_c/l$), where M_K is defined in formula (2.3) and writing the Poisson law under the form

$$h_K(\eta) = \frac{1}{K!}[4\eta]^{K+1} \exp(-4\eta)$$

we are lead to the following set of self consistent equations

$$\begin{cases} z = z_K(\eta) \\ f_K(x) = \dfrac{4}{z^2} \\ h_K(\eta) = \dfrac{4\pi}{x^2 z^2} \end{cases} \tag{2.9}$$

which can be rewritten as an implicit equation in the variable η

$$\frac{4\pi}{[z_K(\eta)]^2 h_K(\eta)} = \left[f_K^{-1}\left(\frac{4}{[z_K(\eta)]^2}\right)\right]^2$$

in such a way that the critical value $\eta_c(K)$ is given by the smallest abcissa corresponding to the intersection points of the curves $F_K(\eta)$ and $H_K(\eta)$ defined by

$$\begin{cases} F_K(\eta) = f_K^{-1}\left(\dfrac{4}{[z_K(\eta)]^2}\right) \\ H_K(\eta) = \sqrt{\dfrac{4\pi}{[z_K(\eta)]^2\, h_K(\eta)}}. \end{cases} \tag{2.10}$$

2.5. Asymptotics

Letting $K \to \infty$, in any location of the border of the spherical $(K+1)$-cluster, there is with probability one a neighbour of the centre. If we consider two such clusters the minimum of the distance between two occupied points is just

$$\lim_{K\to\infty} M_K(l,\delta) = \delta - 2l. \tag{2.11}$$

The corresponding function f_K behaves like $\lim_{K\to\infty} f_K(x) = (x-2)^2$.

In this limit we can study analytically the intersections of the curves $F_K(\eta)$ and $H_K(\eta)$. For sufficiently large K the critical value is found to be

$$\eta_c(K) \approx \frac{1}{4}(K+1)\left(1+\sqrt{\frac{\ln K}{K}}\right). \tag{2.12}$$

2.6. Results and Comments

The functions $f_K(x)$ can be computed by Monte Carlo simulations in order to determine the critical filling factor η_c. (See the set of curves in Figs. 2 and 3). However in the case $K=4$, $f_4(x)$ has been derived analytically and compared to the Monte Carlo simulation (see Appendix and Fig. 6).

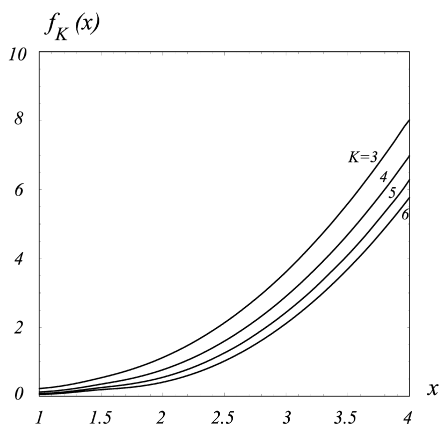

Fig. 2. Functions $f_K(x)$, average square of the minimum distance between spherical $(K+1)$-clusters for $K = 3, 4, 5, 6$.

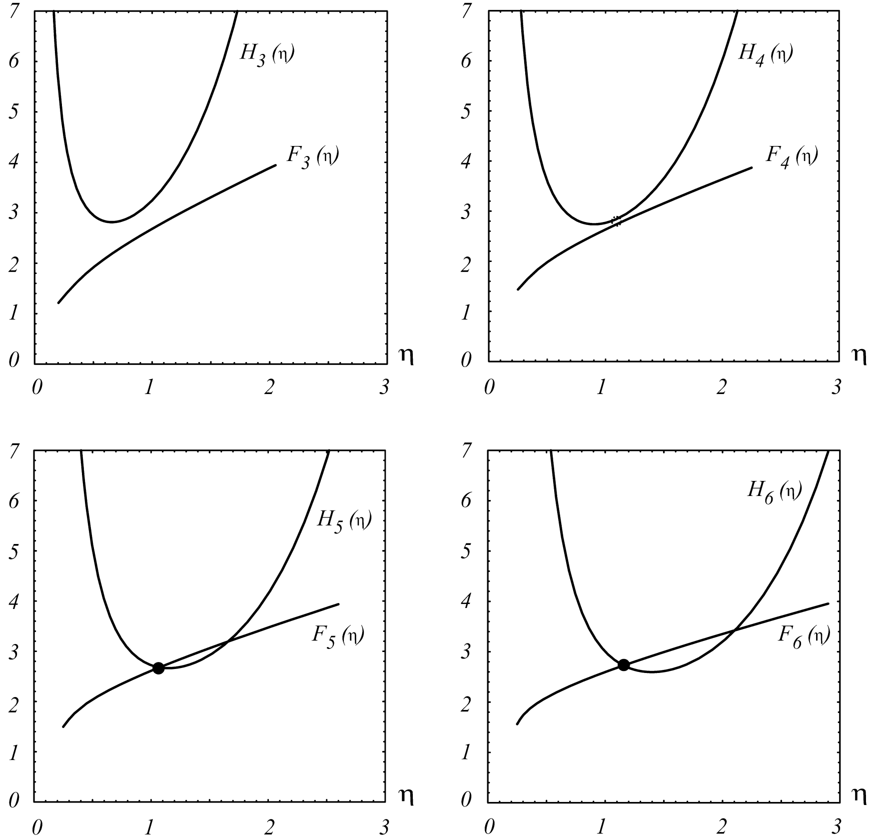

Fig. 3. Percolation critical criterium determined through the intersection of the curves $H_K(\eta)$ and $F_K(\eta)$ for $K = 3, 4, 5, 6$.

The results we get are the following

- there is no solution for $K \leqslant 3$.
- for $K = 4$ the curves are almost tangent but they really cut for $K \geqslant 5$.
- for $K = 5$ or 6 the results are very close and correspond to a value of η_c below 1.2. Surprisingly enough, using any kind of reasonable interpolating procedure between the cases $K = 4$ and $K = 5$ leads to a percolation critical value of

$$\eta_c \approx 1.18$$

which is quite close to the value obtained by numerical simulations.

• for $K > 6$, $\eta_c(K)$ increases and behaves roughly as $\frac{1}{4}(K+1)$ for $K \to \infty$.

In fact percolation seems to appear simultaneously in the different classes of clusters for K between 4 and 6. This justify in some sense the heuristic argument given by Alon and Drory [Al1] concerning the importance near the percolation threshold of occupied points which have at least 4 neighbours. This is also compatible with what can be seen in numerical simulations.

3. GENERALIZATION TO OTHER BASIC SHAPES

One can use our model in order to determine the critical parameter for continuum percolation of objects of various shapes if one postulate the existence of a *mean connectivity distance* between two such objects and moreover, as in the case of disks, consider only "spherical $(K+1)$-clusters."

More precisely, let us consider identical planar objects $\mathscr{P}$ thrown at random in $\mathbb{R}^2$ with density ρ. The size of an object depends on some typical length L. Suppose that the configuration of each of the copies can be characterized by the position x_i of a distinguished point O and a direction specified by an angle φ. The points x_i are distributed according to a Poisson law and the angles are uniformly distributed.

As an example, one can consider needles of a given length $2L$ and take as distinct point O the centre of mass of the needle or one can take squares of a given size, with uniformly distributed orientation or with a fixed orientation.

A point O in space is occupied iff there is an object whose characteristic point is located in O. Two objects (their positions being specified by the doublets $\{O_1, \varphi_1\}$ and $\{O_2, \varphi_2\}$) are adjacent if they overlap. This happens if the distance $|O_1 O_2|$ is less than some distance $2a_\theta$ which depends only of their relative orientation θ.

Definition 3.1. The mean connectivity distance $2a$ between two objects $\mathscr{P}$ is defined as the average

$$2a = \langle 2a_\theta \rangle_\theta \tag{3.1}$$

where θ is a uniformly distributed random variable in a set $\mathfrak{T}$. Two objects are said to be adjacent if $|O_1 O_2| \leqslant 2a$.

Remark 3.1. The mean $a \equiv a(L)$ is proportional to L, the characteristic length of the basic objects $\mathscr{P}$.

As previously we can then define:

• *The radius of the most probable k-cluster*. A k-cluster of objects $\mathscr{P}$ is again a set of k occupied points located in a circle of radius r_0 surrounded by an empty corona $]r_0, r_0+2a]$. As previously (2.2) we obtain the radius of the most probable k-cluster:

$$\tilde{r}_0(\tilde{r}_0+2a)=\frac{k}{\rho\pi}$$

and

$$\frac{\tilde{r}_0(k)}{a}=-1+\sqrt{1+\frac{k}{\rho\pi a^2}}. \tag{3.2}$$

• *Percolation condition*. Using the same scheme defined in the previous Section, we consider two spherical $(K+1)$-clusters of objects $\mathscr{P}$ and we define again the dimensionless parameters l and δ such that

$$|O_1A_i|=la=\tilde{r}_0(K+1)\quad (=|O_2B_j|), \qquad |O_1O_2|=\delta a. \tag{3.3}$$

Using Definition (2.3) for the mean minimum distance $M_K(l, \delta)\, a$ between two $(K+1)$-clusters of objects $\mathscr{P}$, the percolation condition reads

$$M_K(l, \delta)\, a \leqslant 2a \tag{3.4}$$

which gives a critical value δ_c^K of δ, for fixed l

$$M_K(l, \delta_c^K)=2. \tag{3.5}$$

The functions $M_K(1, \delta/l)$ are independent of the peculiar shape of objects considered and the same is true for the percolation condition

$$M_K(1, \delta_c^K/l)=2/l. \tag{3.6a}$$

The second condition giving the mean distance L_K (in a units) between occupied points having K-neighbours (centres of $(K+1)$-clusters) and the density Ξ_K, is also the same

$$(aL_K)^2\, \Xi_K \approx 1$$

where

$$\Xi_K=\rho\mathbb{P}_K(\rho 4\pi a^2)=\rho\,\frac{(\rho 4\pi a^2)^K}{K!}\exp(-\rho 4\pi a^2) \tag{3.6b}$$

and the same is true at the percolation threshold for the condition

$$\delta_c^K \approx L_K. \tag{3.6c}$$

Now, in this framework, we have to use as definition of η, $\eta = \rho\pi a^2$. Then, using (2.9), the smallest intersecting point of the standard curves gives the critical value η_c, whatever percolating objects $\mathscr{P}$ are considered in the generalized model

$$\eta_c = \rho_c \pi a^2 \approx 1.18. \tag{3.7}$$

The specificity of a peculiar percolation problem is then completely contained within the relation $a = a(L)$.

4. EXAMPLES

4.1. Square Percolation

The basic objects are squares of side $2L$ and centre O.

4.1.1. Squares with Random Orientation

We consider one square S_1 (centre O_1) and all other squares (e.g., S_2, centre O_2), with a fixed orientation θ with respect to S_1, in the limit cases for which $S_1 \cap S_2 \neq \varnothing$.

The mean value of $|O_1 O_2|$ can be calculated for θ fixed, as well as the mean value of a for θ uniformly randomly distributed. We obtain

$$2a = 2.42L.$$

Using again

$$\pi \rho_c a^2 = 1.18 = \pi (1.21)^2 \, \rho_c L^2$$

we get

$$\rho_c (2L)^2 = \frac{4(1.18)}{\pi (1.21)^2} = 1.03.$$

This number has to be compared to the values obtained by numerical simulations which are very near 1.10 (see, e.g., [I]).

4.1.2. Squares with Same Orientation

The obtained value here for a is $2a = 2.30L$ and

$$\rho_c(2L)^2 = \frac{4(1.18)}{\pi(1.15)^2} = 1.14$$

which is also in very good agreement with the value obtained by Garboczi (see [G])

$$\rho_c(2L)^2 = 1.147.$$

4.2. Needles Percolation

The basic objects here are needles of length $2L$ and centre O. We consider one needle AB (centre O_1) and all the needles, with fixed orientation θ with respect to AB, which intersect AB.

Their centres have to be within the parallelogram Π and are uniformly distributed. The extreme cases correspond to centres located on the perimeter ∂_Π of Π. Let E be such a centre. Then the mean value $A = \langle |O_1 E| \rangle_{\partial_\Pi}$ is a function of θ. The angle θ is a random variable with uniform distribution and we can calculate the mean value $\langle A \rangle_\theta$ which has to be identified with $2a$. We obtain

$$2a = 1.09L.$$

The percolation critical parameter $\rho_c(2L)^2$ for needles of length $2L$ is known from numerical simulations: $\rho_c(2L)^2 = 5.71 \pm 0.12$. Using (3.7), our model leads to

$$\pi \rho_c a^2 = 1.18 = \pi (0.543)^2 \, \rho_c L^2$$

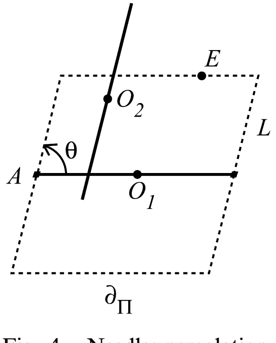

Fig. 4. Needles percolation.

and

$$\rho_c(2L)^2 = \frac{4(1.18)}{\pi(0.543)^2} = 5.08.$$

This can be considered as a good approximation with regards to the great generality of the model and the extreme geometric features of needles as percolating objects.

5. CONCLUSION

We have discussed a model for continuous percolation in $\mathbb{R}^2$ for quite general geometrical object shapes based on probabilistic estimations of events (k-clusters) describing both the most probable shape and the conditions for (local) connectivity of these clusters at criticality. This model leads to critical percolation parameters in fairly good agreement with both numerical computations and heuristic criteria proposed so far. Moreover it gives a hint for the structure of the connected clusters near the percolation threshold.

The study of the scaling properties of spherical $(K+1)$-clusters near criticality through a renormalization group (RG) technique is under investigations. Indeed the simple geometric features of the objects used in the model should allow a manageable application of RG theory to see whether the simple local arguments that have been developed in this work concerning critical percolation densities can also lead to reasonable scaling predictions.

We have tried to present our arguments in such a way as to stress the basic ideas and approximations.

APPENDIX. AVERAGE MINIMUM DISTANCE BETWEEN DISKS OF TWO SPHERICAL $(K+1)$-CLUSTERS

The square distance between the disks A_i, B_j of two spherical $(K+1)$-clusters is given by (see Section 2.2 and Fig. 5)

$$\lambda_K(x) = [x - \cos(\varphi) - \cos(\theta)]^2 + [\sin(\varphi) - \sin(\theta)]^2 \tag{5.1}$$

with $x = \delta/l = |O_1O_2|/|O_1A_1|$.

We want to compute the following average

$$\begin{aligned} f_K(x) = [M_K(1, x)]^2 &= \langle \min_{(i,j)} \{\lambda_K(A_i, B_j, x)_{i,j\in I_K}\}\rangle_{\varphi_1,\dots,\varphi_K,\theta_1,\dots,\theta_K} \\ &= \langle m_K(x)^2\rangle_{\varphi_1,\dots,\varphi_K,\theta_1,\dots,\theta_K}. \end{aligned} \tag{5.2}$$

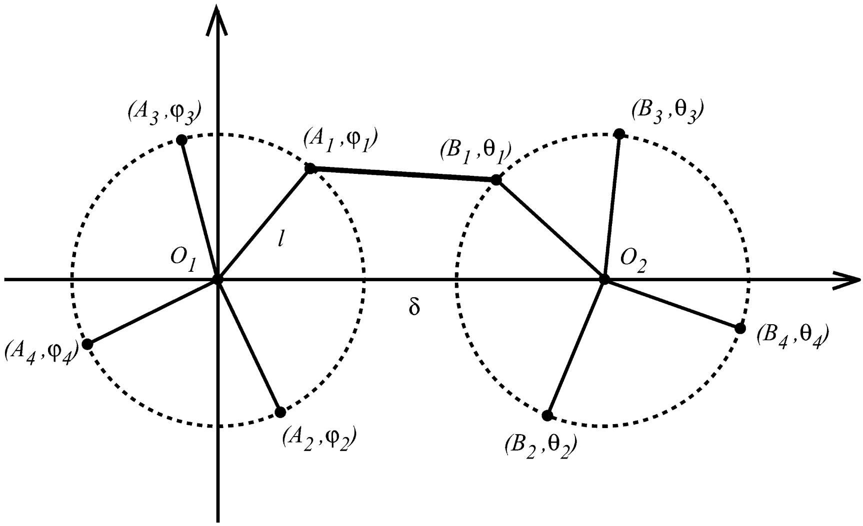

Fig. 5. Distance between spherical $(K+1)$-clusters.

φ and θ are uniformly distributed random variables in $[0, 2\pi]$. Introducing random variables α and β (with the same distribution) such that $\varphi = \alpha + \beta$ and $\theta = \alpha - \beta$, one gets

$$\lambda_K^{\alpha, \beta}(x) = 2 + x^2 + 2\cos(2\alpha) - 2x[\cos(\alpha+\beta) + \cos(\alpha-\beta)]. \tag{5.3}$$

This expression is invariant under sign symmetry in both α and β variables. It allows us

— first to consider the (φ_i, θ_j)-averages only on the upper half-clusters,

— thence, to suppose both the A_i's and B_j's to be ordered on each upper half cluster.

From this it follows that $m_K(x)^2 = \min_{(i,j)}\{\lambda_K(A_i, B_j)_{i,j \in I_K}\}$ is obviously the distance between the closest points from each (half) spherical cluster. The average reduces to

$$\left(\frac{K!}{\pi^K}\right)^2 \int_0^\pi d\varphi_1 \int_0^{\varphi_1} d\varphi_2 \cdots \int_0^{\varphi_{K-1}} d\varphi_K \int_0^\pi d\theta_1 \int_0^{\theta_1} d\theta_2 \cdots \int_0^{\theta_{K-1}} d\theta_K \, m_K(x)^2 \tag{5.4}$$

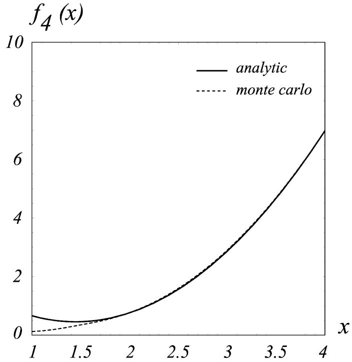

Fig. 6. Comparison of computed and simulated average minimum distance $f_4(x)$ between disks belonging to two spherical ($K=4$)-clusters.

where $K!$ is the number of ways one can order the K disks on each half spherical cluster. We find, for $K=4$

$$f_4(x) = 2 + x^2 + \frac{4608}{\pi^8} - \frac{3456}{\pi^6} + \frac{96(7+2x)}{\pi^4} - \frac{16(2+3x)}{\pi^2} \tag{5.5}$$

In Fig. 6 we have represented both the results of a Monte Carlo simulation and the function (5.5) $f_4(x)$ showing an excellent agreement provided $\delta \geqslant 2l$ which is the distance below which the spherical 4-clusters begin to overlap.

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